# ACCURATE ESTIMATION OF PEAK VON MISES STRESS AND COMPOSITE FAILURE METRICS IN RANDOM SIMULATION

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## 1. Introduction

Random vibrations are non-deterministic in nature and can be simulated using probabilistic approaches. Random vibration standards exist in the automotive industry to protect subsystem mechanical designs against failures caused by road roughness; similarly, the space industry imposes random vibration base acceleration specifications on components such as electronic units and antennas, in order to validate their mechanical design with respect to structure and acoustic-borne random vibrations.

The probabilistic nature of random vibrations requires the determination of confidence levels appropriate for equipment validation. A common industrial practice sets the confidence at 99.73%, meaning that during a random event such as a test, the design or peak loads are allowed to be exceeded 0.27% of the time. For a zero-mean Gaussian process, this 99.73% confidence corresponds to levels that are 3 times the root-mean-square (rms) level: The typical ‘3-rms’ industrial approach consists in extracting rms responses from numerical simulations and multiplying them by a factor of 3.

Gaussian processes remain Gaussian when subjected to linear operators: If the input loads on a structure are Gaussian, so is its response. However, this is not true of certain derived responses, such as Von Mises stress: Whereas stress tensor components follow Gaussian probability distributions, the quadratic Von Mises stress equation produces non-zero mean results that observe non-Gaussian distributions. Hence, for Von Mises stress and other derived response quantities like composite Tsai-Wu failure indices that have arbitrary and variable probability distributions, the ‘3-rms’ approach produces peak results that correspond to unknown confidence levels.

Finite element programs must be used to compute the responses of reasonably complex structures to random inputs. MAYA's random processor evaluates peak responses as a function of user-defined confidence levels, including key derived quantities such as Von Mises stress and Tsai-Wu failure metrics.

This paper compares the numerical approximation of peak Von Mises stress and composite failure metric quantities with Monte Carlo simulations, and establishes error estimates for the traditional '3-rms' approach using an industrial finite element model.

## 2. Finite Element Model

The space antenna model used in this study consists of 13,500 beam, shell and solid elements, as well as 8,300 nodes. The antenna main reflectors, inter-reflector structure, sub-reflector and struts are fabricated from kevlar and carbon fiber composites. The central feed structure is metallic, and its elements were used for the Von Mises stress calculations, while the laminate elements were used for Tsai-Wu metrics.

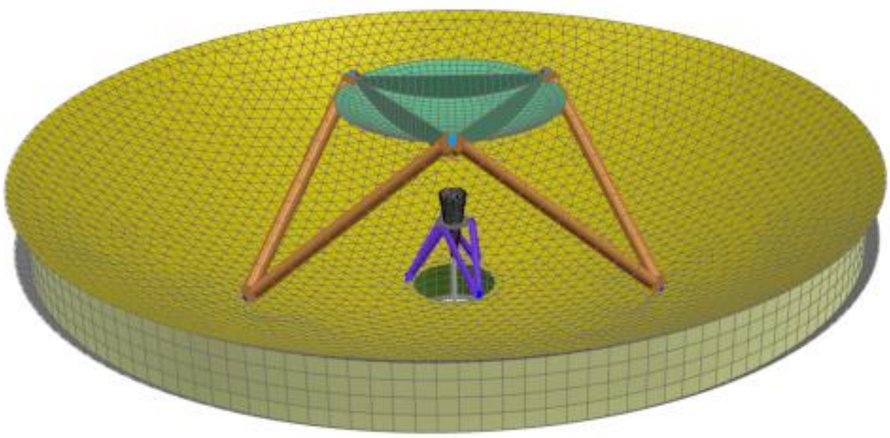


*Figure 1: Space Antenna Model*

A base acceleration was applied in the lateral directions at a central node connected rigidly to the antenna base structure. Its power spectral density was defined by the curve in Figure 2.

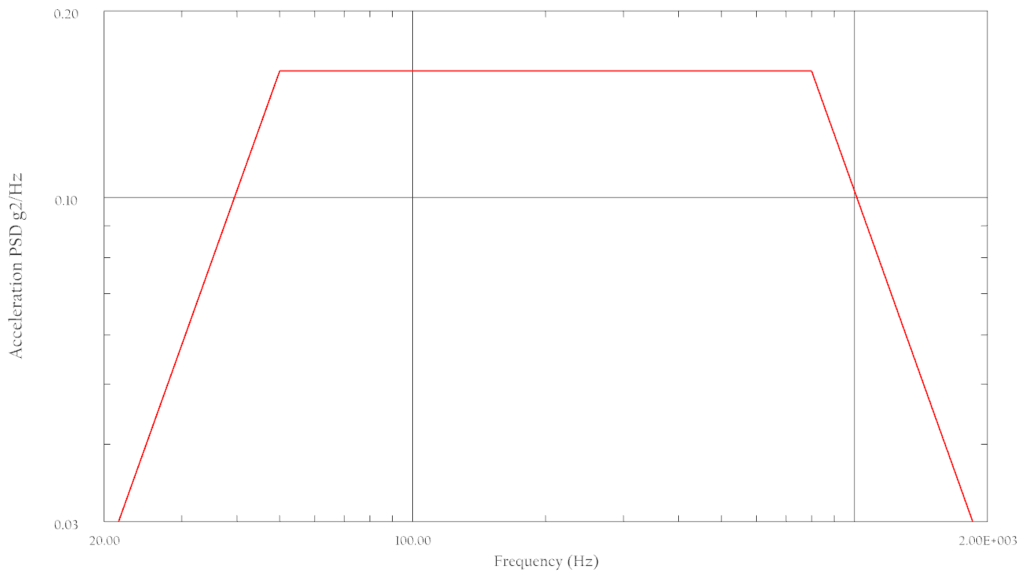



*Figure 2: Base Acceleration power spectral density*

The first 10 constrained normal modes were extracted with the NX Nastran solver and imported into the random processor.

## 3. Monte Carlo Simulation for 2D Elements

### Obtaining samples of $\sigma_x$, $\sigma_y$, $\sigma_{xy}$

In order to perform the Monte Carlo simulation, we first need *n* random samples of Gaussian zero-mean cartesian stresses ($\sigma_x$, $\sigma_y$, $\sigma_{xy}$) which are consistent with the symmetric variance-covariance matrix of the cartesian stresses [S] obtained from the random processor and computed using the method described in [2]. Once the matrix [S] is available, it is supplied as the SIGMA argument along with a mean vector MU of 0.0 to the MATLAB© function “mvnrnd“ [3], which then generates the required multivariate normal random samples.

### Calculating Von Mises stress

For 2D elements with isotropic material properties, the Von Mises stress $\sigma_{vm}$ was computed for all of the ($\sigma_x$, $\sigma_y$, $\sigma_{xy}$) samples using Equation (1).

$$\sigma_{vmi}^2 = \begin{bmatrix}\sigma_{xi} & \sigma_{yi} & \sigma_{xyi}\end{bmatrix}\begin{bmatrix}V\end{bmatrix}\begin{pmatrix}\sigma_{xi} \\ \sigma_{yi} \\ \sigma_{xyi}\end{pmatrix} \qquad i = 1,\ldots,n \qquad (1)$$

Where $[V]=\begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (2)

**Statistics**

Statistics for the population of Von Mises stresses were then computed, such as mean (4), variance (5) and rms (6), where $x_i = \sigma_{vmi}$. The cumulative distribution function was found from Equation (7), and the probability density function was found by numerically differentiating this cumulative distribution function (central difference method) via Equation (8).

$x = [x_{1,}, x_{2,},..., x_n]^T$ , $x^s = [x_1^s, x_2^s,..., x_n^s]^T$ such that $x_1^s \le x_2^s \le ... \le x_n^s$ (3)

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (4)$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu)^2 \quad (5)$$

$x_{rms} = \sqrt{\sigma^2 + \mu^2}$ where σ is the standard deviation (6)

$P(X \le x) = \sum_{x_i \le x} p(x_i) = \frac{n_{x_i \le x}}{n}$ , $i = (1,..,n)$ and $x \in \Re$ (7)

$p(x_i) = \frac{P(X \le x_{1+1}) - P(X \le x_{1-1})}{x_{i+1} - x_{i-1}}$ , $i = (2,..,n-1)$ (8)

**Peak**

The peak Von Mises stress corresponding to the chosen confidence level is defined by (9), where $P(X \le x)$ is the cumulative distribution function of the Von Mises stress. The algorithm did this by sorting the Von Mises samples in ascending order ($\sigma_{vmi}{}^s$), and choosing the sample corresponding to *conf* as in Equation (10). The peak-to-rms ratio was then calculated by Equation (11).

$P(X \le x_{peak}) = conf$ , $0 \le conf \le 1$ (9)

$\sigma_{vm,peak} \cong \sigma_{vmi}^s |_{i=p}$ , $p = ceil(conf \cdot n)$ (10)

$$ratio_{vm} = \frac{\sigma_{vm,peak}}{\sigma_{vm,rms}} \tag{11}$$

## 4. Calculating Composite Failure Metrics

For 2D laminate composite elements, ply stresses are on-axis: For a given unidirectional ply of a 2D element, $\sigma_x$ corresponds to the fiber direction, $\sigma_y$ corresponds to the matrix direction and $\sigma_{xy}$ represents in-plane shear. The only difference for a ply of woven material is that $\sigma_y$ corresponds to the weft fiber direction. Both the Tsai-Wu failure indices ($FI_i$) and strength ratios ($\alpha_i$) were found for each stress sample $\sigma_{xi}$, $\sigma_{yi}$, $\sigma_{xyi}$.

### Statistics

Calculation of the mean, variance and rms of $FI_i$ and $\alpha_i$ was performed using Equations (4) through (6) where $x_i$ was replaced by either $FI_i$ and $\alpha_i$. Note that this calculation was performed for each ply of each element, since each ply has its own distribution of failure indices and strength ratios. Similarly, the cumulative distribution function and probability density functions were obtained from Equations (7) and (8), respectively.

### Peak

The peak failure index $FI_i$ was calculated in a manner identical to the Von Mises stress; failure indices were sorted in increasing order into $FI_i^s$, with $FI_{peak}$ given by Equation (12). For strength ratios $\alpha_i$, the values were sorted in descending order and therefore a minimum rather than a peak was found, see Equation (13). The peak-to-rms ratio for $FI_i$ and rms to minimum ratio for $\alpha_i$ were also calculated, see Equations (14) and (15).

$$FI_{peak} \cong FI_p^s,\ p = ceil(conf \cdot n) \tag{12}$$

$$\alpha_{\min} \cong \alpha_m^{s,descending},\ m = ceil(conf \cdot n) \tag{13}$$

$$ratio_{FI} = \frac{FI_{peak}}{FI_{rms}} \tag{14}$$

$$ratio_{\alpha} = \frac{\alpha_{rms}}{\alpha_{\min}} \tag{15}$$

## Tsai-Wu Failure Criterion in 2D

For each ply stress sample, the Tsai-Wu failure index $FI_{Tsai-Wu}$ was calculated using Equation (16). The constants are given by Equations (17) through (19). Note that $F_{12}$ was considered as a material property defined in the finite element model.

$$FI_{Tsai-Wu} = F_{11}\sigma_{xi}^2 + F_{22}\sigma_{yi}^2 + 2F_{12}\sigma_{xi}\sigma_{yi} + F_{44}\sigma_{xyi}^2 + F_1\sigma_{xi} + F_2\sigma_{yi}\ ,\ i = 1,...,n \quad (16)$$

$$F_{11} = \frac{1}{X_t X_c}\ ,\ F_{22} = \frac{1}{Y_t Y_c} \quad (17)$$

$$F_{44} = \frac{1}{S^2} \quad (18)$$

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}\ ,\ F_2 = \frac{1}{Y_t} - \frac{1}{Y_c} \quad (19)$$

## Strength Ratio

For each $\sigma_{xi}$, $\sigma_{yi}$, $\sigma_{xyi}$ sample, the strength ratio $\alpha_i$ was calculated by multiplying each stress component in Equation (16) by $\alpha_i$ and setting the failure index to 1, leading to Equation (20). This quadratic equation was then solved for $\alpha_i$, taking the smallest positive value. The strength ratio was directly solved for using Equation (23), provided $A_i$ was not very near or equal to zero, in which case the iterative method of Equation (25) was used to avoid large rounding error.

$$0 = F_{11}\alpha_i^2\sigma_{xi}^2 + F_{22}\alpha_i^2\sigma_{yi}^2 + 2F_{12}\alpha_i\sigma_{xi}\alpha_i\sigma_{yi} + F_{44}\alpha_i^2\sigma_{xyi}^2 + F_1\alpha_i\sigma_{xi} + F_2\alpha_i\sigma_{yi} - 1\ ,\ i = 1,...,n \quad (20)$$

$$0 = \alpha_i^2\left(F_{11}\sigma_{xi}^2 + F_{22}\sigma_{yi}^2 + F_{44}\sigma_{xyi}^2 + 2F_{12}\sigma_{xi}\sigma_{yi}\right) + \alpha_i\left(F_1\sigma_{xi} + F_2\sigma_{yi}\right) - 1\ ,\ i = 1,...,n \quad (21)$$

$$0 = \alpha_i^2 A_i + \alpha_i B_i - 1,\ i = 1,...,n \quad (22)$$

$$if A_i \geq 10^{-6}, \alpha_{+,-} = \frac{-B_i \pm \sqrt{B_i^2 + 4A_i}}{2A_i}\ ,\ \alpha_i = \left[\max(1/\alpha_+, 1/\alpha_-)\right]^{-1}, i = 1,...,n \quad (23)$$

$$if A_i \leq 10^{-6}, F_i(x) = A_i x^2 + B_i x - 1\ ,\ F_i^{'}(x) = 2A_i x + B_i\ ,\ i = 1,...,n \quad (24)$$

While $abs(F_i(x_k)) \geq tol$ , $x_{k+1} = x_k - \dfrac{F_i(x_k)}{F_i^{'}(x_k)}$ , $\alpha_i = x_k$ (25)

## 5. Accuracy of Monte Carlo Simulation

The accuracy of the Monte Carlo simulation results depends on the number of samples used, *n*, and the desired probability, $p_f$. The probability is related to the confidence level as follows:

$$p_f = 1 - conf \tag{26}$$

For all results presented here, $n = 5 \times 10^6$ was used for a balance of accuracy and computation time. Any two simulations will not return the *exact* same value for the peak ($\sigma_{vm,peak}$, $FI_{peak}$ or $\alpha_{min}$), unless *n* is extremely large or unless $p_f$ is relatively high. So a distribution of peak values could be obtained from running many simulations, and ideally each simulation would return nearly the same peak. The coefficient of variation $COV = \sigma / \mu$, measures dispersion: a lower coefficient corresponds to a more accurate simulation. A coefficient of less than 10% is recommended [2] for peak failure metrics from Monte Carlo simulation.

The coefficient of variation for the estimated peak is given by Equation (27)[2] and is shown in Table 1 for several $p_f$ values and for $n$ = 5 million samples. Table 1 suggests the number of samples chosen is suitable for all values of $p_f$, except the lowest, for which $1.75 \times 10^8$ samples would be required to achieve a 10% coefficient of variation.

$$COV(p_f) = \frac{1}{p_f}\sqrt{\frac{(1-p_f)p_f}{n}} \tag{27}$$

**Table 1: Coefficient of Variation as a function of confidence level**

| Confidence | $p_f$ | COV (%) |
|---|---|---|
| 0.99 | $1.00 \times 10^{-2}$ | 0.4 |
| 0.9973 | $2.70 \times 10^{-2}$ | 0.9 |
| 0.99994 | $6.33 \times 10^{-5}$ | 5.6 |
| 0.9999994 | $5.73 \times 10^{-7}$ | 59.1 |

In the following sections, we will determine the error inherent in assuming a Gaussian distribution for non-Gaussian responses such as

Von Mises stress and Tsai-Wu failure metrics: We will do this by comparing the Monte Carlo peak-to-rms ratio to the equivalent Gaussian peak-to-rms ratio *r* as per equations (28) and (29).

$$Conf = \int_{-r\sigma}^{r\sigma} P dx = \int_{-r\sigma}^{r\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} dx = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]_{-r\sigma}^{r\sigma} = \mathrm{erf}\left(\frac{r}{\sqrt{2}}\right) \qquad (28)$$

Where σ is the standard deviation and is equivalent to the rms when the mean μ is zero. The Gaussian peak-to-rms ratio *r* is such that:

$$peak = r * rms \qquad (29)$$

For a 99.73% confidence level, the Gaussian peak-to-rms ratio *n* is 3.

## 6. Results

### Von Mises Stress

The Monte Carlo simulation results for all 1056 2D metallic elements in the antenna feed are presented in Table 2. The Von Mises stress distribution was calculated at the top and bottom of each element. The percent error refers to the error in assuming the Von Mises stress is Gaussian, given by Equation (30). The minimum peak-to-rms ratio (Min Ratio) and the maximum peak-to-rms ratio (Max Ratio) both refer to $ratio_{vm}$ computed from Equation (11).

**Table 2: Von Mises stress results for antenna model**

| Confidence (%) | r | Axis | Min Ratio | Error (%) | Axis | Max Ratio | Error (%) |
|---|---|---|---|---|---|---|---|
| 99.00 | 2.5758 | Y | 2.055 | 25.32 | Y | 2.574 | 0.07 |
| 99.73 | 3 | Y | 2.326 | 28.99 | Y | 2.999 | 0.04 |
| 99.994 | 4 | Y | 2.970 | 34.66 | Y | 4.004 | -0.09 |
| 99.9999 | 5 | Y | 3.614 | 38.36 | Y | 5.105 | -2.05 |

$$Error(\%) = \frac{\sigma_{vm,peak} - r\sigma_{vm,rms}}{\sigma_{vm,peak}} X100 = \frac{ratio_{vm} - r}{ratio_{vm}} X100 \qquad (30)$$

Figure 3 shows the cumulative distribution function CDF and probability density function PDF of the Von Mises stress for the element with the minimum $ratio_{vm}$ at 3-rms (99.73%) confidence level.

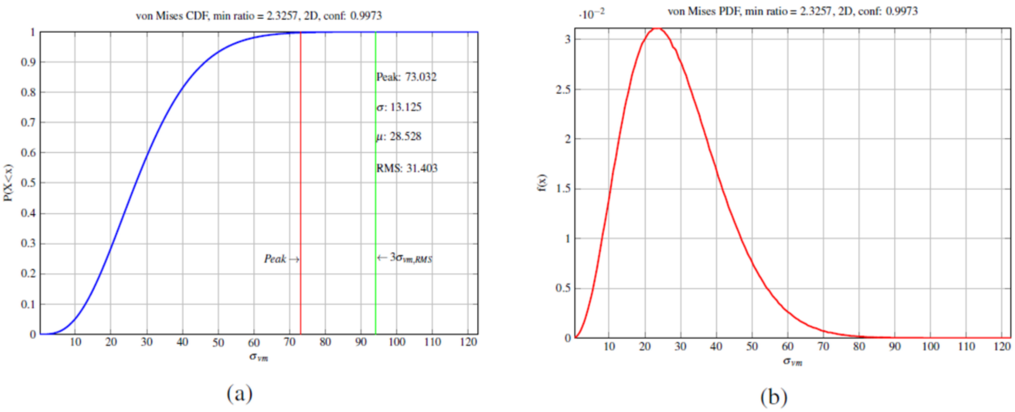



*Figure 3: CDF and PDF of Von Mises stress for minimum ratio*

Figure 4 shows the CDF and PDF of the Von Mises stress for the element with the maximum $ratio_{vm}$ at 3-rms confidence level.

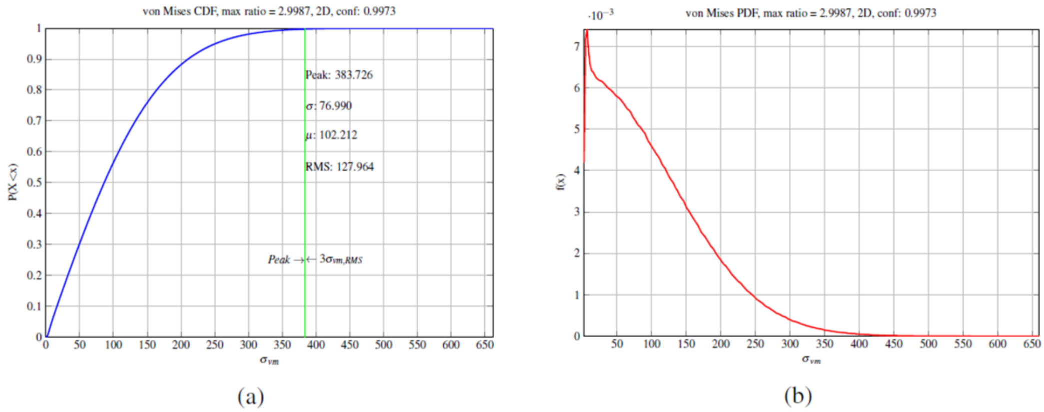



*Figure 4: CDF and PDF of Von Mises stress for maximum ratio.*

Figure 5 compares the cumulative distribution function of $ratio_{vm}$ obtained by the random processor and the Monte Carlo simulation at 3-rms (99.73%) confidence level: It can be seen that the curves are almost identical.

Figure 5 also shows that 20% of elements have a peak-to-rms ratio below 2.65, and 50% of the elements in the model have a peak-to-rms ratio below 2.85: For all these elements, and for 99.73% confidence, the 3-rms assumption overestimates the Von Mises stresses, and hence Equation (29) seems to be conservative. Table 2 shows this is not the case for the two highest confidence levels, as the Gaussian assumption underestimates the true peak by 0.09% and 2.05%, respectively. The same table shows that as the confidence level increases, so does the error in the minimum peak-to-rms ratio. However, the peak Von Mises stress estimated by Monte Carlo simulation also becomes less accurate as reliability increases, based on the COV.

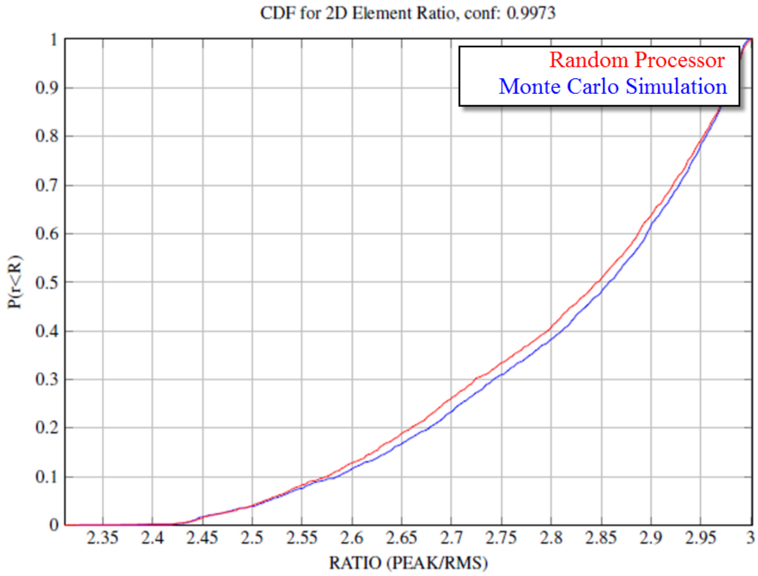



*Figure 5: CDF of the peak-to-rms ratio for the Von Mises stress at 99.73% confidence.*

The max ratio PDF in Figure 4b looks much more like a bell curve than the PDF of the min ratio shown in Figure 3b: In the max ratio case, the peak Von Mises stress is almost exactly the same as if the Von Mises stress were Gaussian. Note the similarity of the CDF for the max ratio with the CDF of a one-sided standard normal distribution shown in Figure 6.

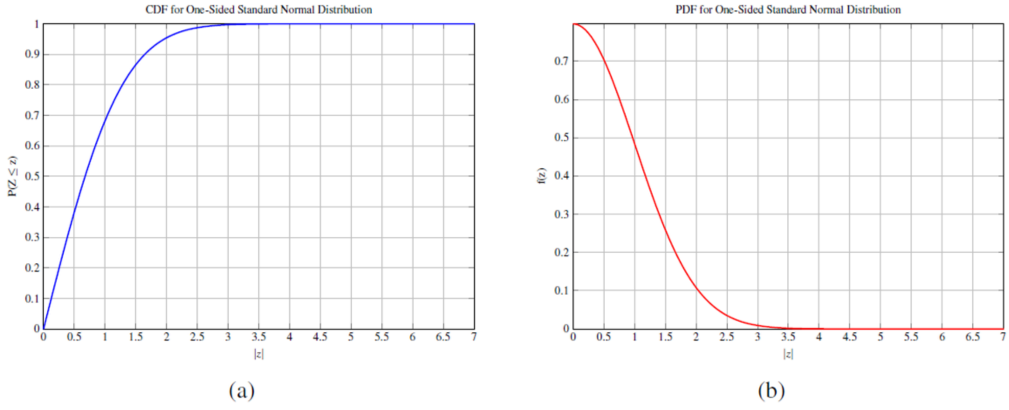



*Figure 6: CDF and PDF of absolute value of a standard normal random variable.*

It can be shown that the Von Mises stress sometimes approaches the absolute value of a standard normal random variable times $\sigma_{vm,RMS}$, which makes $\sigma_{vm,peak}$ match the Gaussian approximation exactly.

## Tsai-Wu Failure Criterion

A summary of results for the Tsai-Wu failure criterion is given in Table 3 for failure index FI and in Table 4 for the strength ratio α.

**Table 3: Tsai-Wu failure index results**

| Confidence (%) | r | Axis | Min Ratio | Error (%) | Axis | Max Ratio | Error (%) |
|---|---|---|---|---|---|---|---|
| 99.00 | 2.5758 | Y | 3.276 | -21.37 | Y | 3.835 | -32.84 |
| 99.73 | 3 | Y | 4.220 | -28.91 | Y | 5.210 | -42.42 |
| 99.994 | 4 | Y | 6.984 | -42.73 | Y | 9.265 | -56.83 |
| 99.9999 | 5 | Y | 10.686 | -53.21 | Y | 15.869 | -68.49 |

The percent error in Table 3 is computed using Equation (31) and using the number of Gaussian rms values *r*. Min and Max Ratio refer to $ratio_{FI}$ of Equation (14).

$$Error(\%) = \frac{ratio_{FI} - r}{ratio_{FI}} X100 \quad (31)$$

**Table 4: Tsai-Wu strength ratio results**

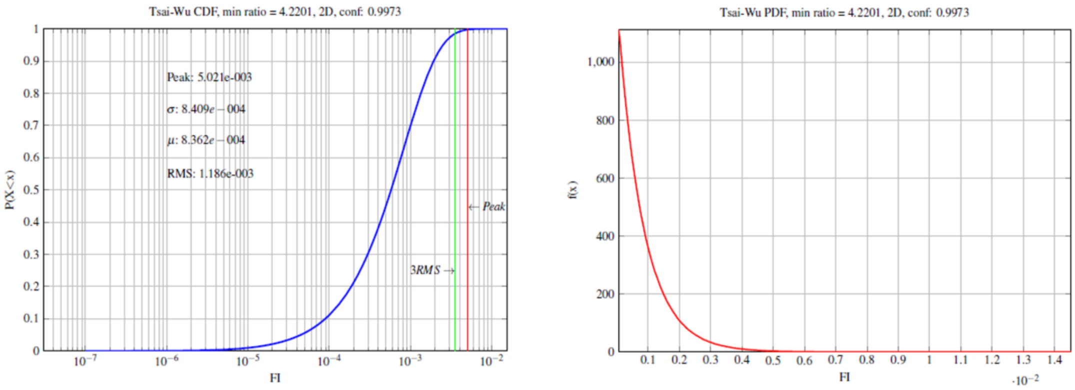
| Confidence (%) | r | Axis | Min Ratio | Error (%) | Axis | Max Ratio | Error (%) |
|---|---|---|---|---|---|---|---|
| 99.00 | 2.5758 | Y | 5.288 | -51.29 | Y | 474.619 | -99.46 |
| 99.73 | 3 | Y | 6.037 | -50.31 | Y | 577.671 | -99.48 |
| 99.994 | 4 | Y | 7.929 | -49.55 | Y | 1029.217 | -99.61 |
| 99.9999 | 5 | Y | 9.726 | -48.59 | Y | 990.665 | -99.50 |

The percent error in Table 4 is computed using Equation (32).

$$Error(\%) = \frac{ratio_{\alpha l} - r}{ratio_{\alpha}} X100 \quad (32)$$

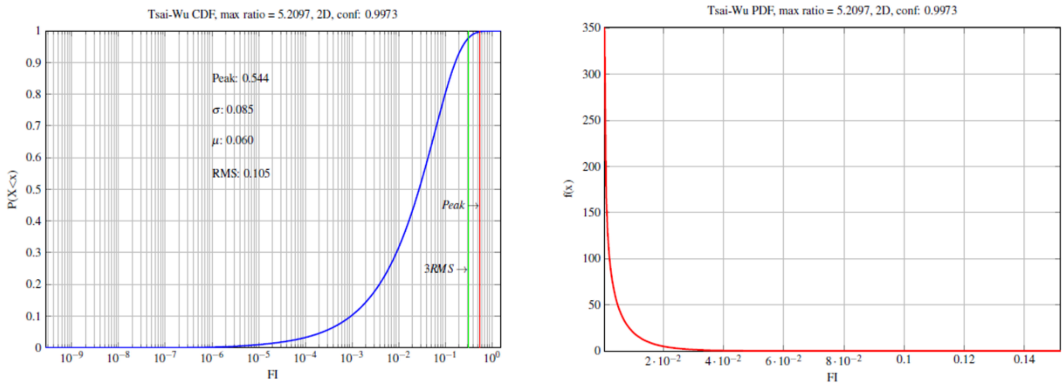
In Table 5, the minimum peak-to-rms ratio Min Ratio and the maximum peak-to-rms ratio Max Ratio refer to $ratio_\alpha$ of Equation (15). Assuming that the Tsai-Wu failure index and strength ratio are Gaussian is clearly erroneous – this assumption is *never* conservative for either result. Like for the Von Mises stress, this assumption gets worse at higher confidence levels.

Figures 7 and 8 plot the cumulative distribution function CDF and probability density function PDF of the Tsai-Wu failure index for the elements with the minimum and maximum $ratio_{FI}$, respectively.




(a) (b)

*Figure 7: CDF and PDF of Tsai-Wu failure index for min ratio*




(a) (b)

*Figure 8: CDF and PDF of Tsai-Wu failure index for min ratio*

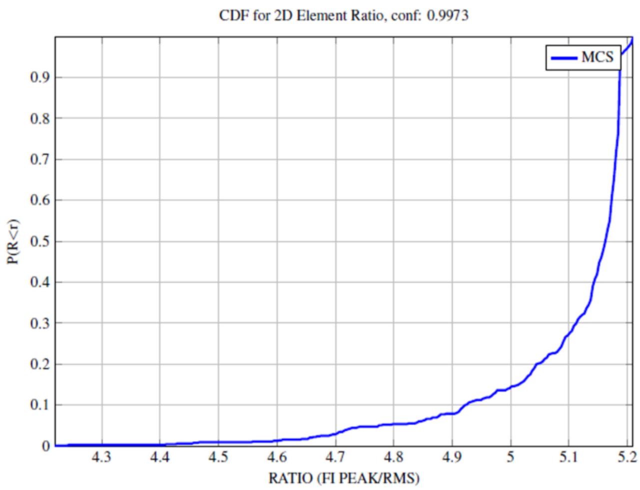



*Figure 9: CDF of the peak-to-rms ratio for the Tsai-Wu failure index at 99:73% probability, Monte Carlo Simulation*

Figures 10 and 11 plot the CDF and PDF of the Tsai-Wu strength ratio for the elements with the minimum and maximum $ratio_{\alpha}$, respectively.

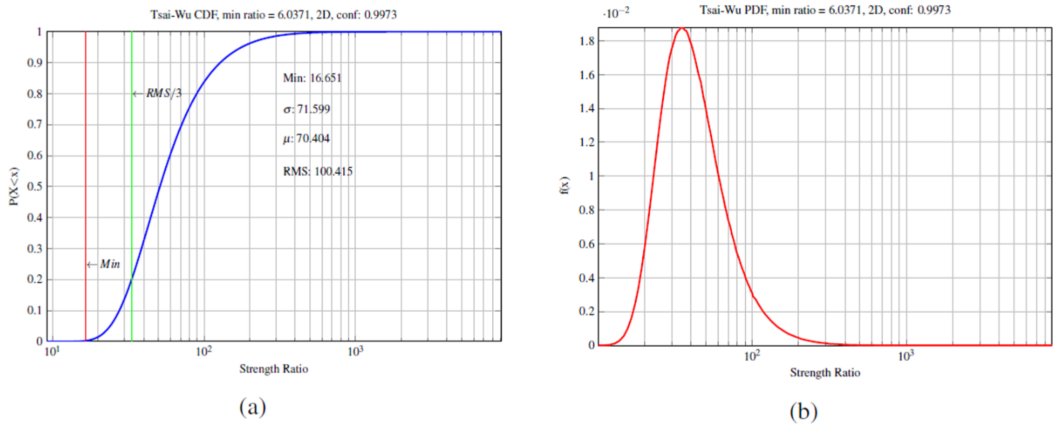



*Figure 10: CDF and PDF of Tsai-Wu strength ratio for min ratio*

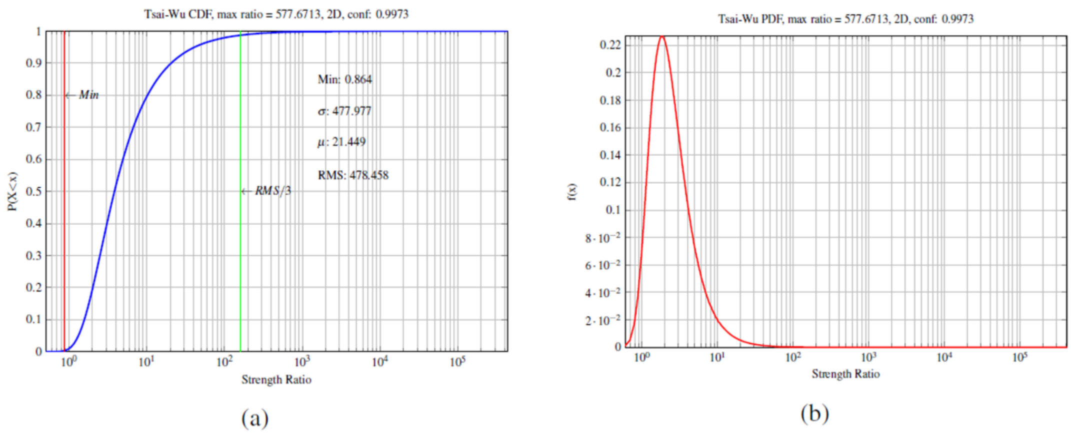



*Figure 11: CDF and PDF of Tsai-Wu strength ratio for max ratio*

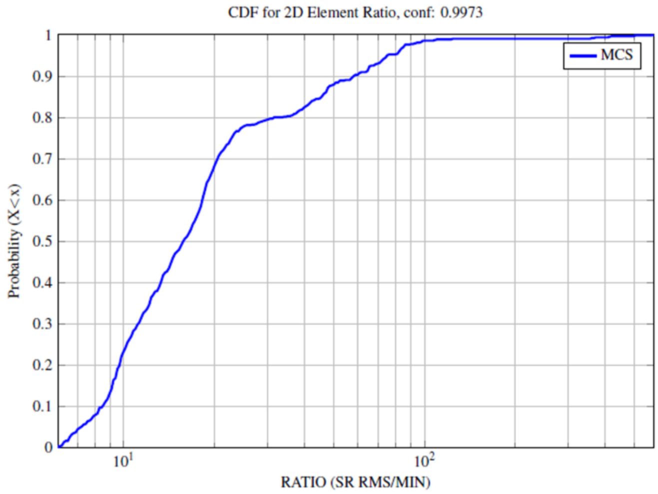



*Figure 12: CDF of the rms-to-min ratio for the Tsai-Wu strength ratio at 99.73% probability, Monte Carlo simulation*

The CDF of $ratio_{FI}$ and $ratio_{\alpha}$ is plotted in Figures 9 and 12. Both plots are for a 99.73% confidence level. Note that when ply stresses are low, the failure index becomes very small, and so the CDF and PDF become skewed towards very small numbers. Rounding error with small numbers makes it difficult to differentiate the CDF and adequately plot the PDF when in this case. Similarly, the strength ratio is skewed towards very large numbers.

## 7. Performance

The random processor is parallelized, runs in batch and features advanced integration algorithms that don't require the user to guess at the integration frequencies. As a result, it allows for the accurate and efficient solution of large models that previously would not have been feasible. For example, Figure 13 shows a finite element model consisting of 600,000 solid elements and 980,000 nodes. Extraction of peak Von Mises stresses in 3 axes and over 250 modes took 120 minutes on a Windows desktop with 48 GB of RAM.

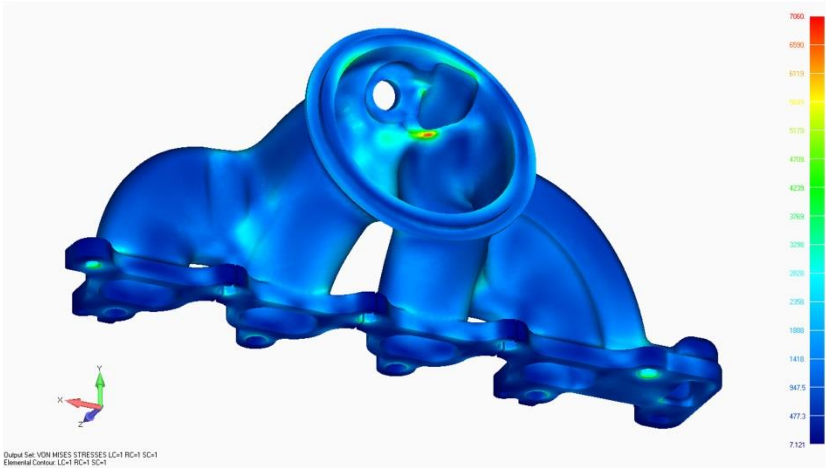



*Figure 13: Large manifold model*

## 8. Software

The random processor described herein is available in the NX Laminate Composites module that is an add-on to the Siemens NX Advanced FEM package, as well as in MAYA's standalone SAToolkit professional software.

## 9. Conclusion

This paper demonstrates the importance of calculating exact failure metric statistics in a random vibration analysis.

For the antenna model, the Gaussian assumption for Von Mises stress is conservative by as much as 29.1% at 99.73% confidence level, in comparison to the exact peak value. This assumption is more conservative for higher reliabilities, increasing to as much as 38.36% at 5$\sigma$ (99.9999% confidence), while mostly underestimating the Von Mises stress. Calculating the exact, peak Von Mises stress could therefore significantly reduce conservatism.

Assuming that either the Tsai-Wu failure index or strength ratio are Gaussian proves to be erroneous, and the error inherent in this assumption is large: For 99.73% confidence, the error is *at least* 29.04% for the failure index and 50.31% for the strength ratio.

Finally, by coupling accurate peak prediction algorithms with high-performance integration schemes, reduced domain integration and parallelization, it is possible to confidently and efficiently process large finite element models and modal bases that until recently were deemed impractical: As a result, structural analysts can spend less effort abstracting and idealizing geometry by using larger, possibly solid models, and can simulate entire systems rather than single components.

## 10. References

[2] Chen, M.-T., & Harichandran, R. (1998). Statistics of the Von Mises stress response for structures subjected to random excitations. *Shock and Vibration*, *5*, 13-21.

[3] MATLAB online documentation for routine mvnrnd: http://www.mathworks.com/help/stats/mvnrnd.html